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THE DESIGN OF CAVITATION-FREE HYDROPROFILES BASED ON THE SOLUTION OF A MIXED INVERSE BOUNDARY-VALUE PROBLEM†

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It is proposed to modify the suction surface of a wing profile in such a way as to reduce the maximum velocity and to exclude cavitation. In mathematical terms, the problem is reduced to a mixed inverse boundary-value problem in aerodynamics, with part of the profile contour prescribed and the other part to be determined so as to obtain a pre-assigned velocity distribution on it. A numerical–analytical method is proposed and elaborated and a computer algorithm is outlined for the design and optimization of cavitation-free profiles. Examples of computed profiles are given.

It is well known that, at positive angles of attack, cavitation may occur on the upper suction surface of a hydroprofile (a hydrofoil or propeller blade) due to low pressure, i.e. high velocities on the surface. Under certain circumstances the effect may be avoided by specially designing the shape of the suction surface so as to limit the magnitude of the velocity. Taking the model of an ideal incompressible fluid, one then arrives at a mixed inverse boundary-value problem for analytic functions [1]. On the assumption that the entire contour is impermeable, the velocity distribution υ is usually specified on the unknown part of the contour, as a function of one of the parameters: s—the arc-length, x—the abscissa, and γ—a parameter in the canonical region. To solve the problem one must complete the known part of the hydroprofile and determine its hydrodynamic characteristics. The earliest publications in this area, which dealt with the design of aerofoils, are apparently [2–4] (see also [5, pp. 448–456]). A survey of later publications relevant to mixed inverse boundary-value problems may be found, for example in [6].

In this paper it is assumed that the unknown section of the hydroprofile is to be determined from the velocity distribution υ as a function of the arc-length s, where $\upsilon(s)$ is specified so as to exclude cavitation.

1. STATEMENT OF THE PROBLEM

An ideal incompressible fluid is flowing smoothly around an impermeable wing profile with a smooth contour L and sharp trailing edge B in the plane $z=x+iy$ The flow is assumed to have a velocity $\upsilon_\infty e^{i\alpha}$ at infinity (Fig. 1a). The contour L is divided by a given point C into a known part (L_1) and an unknown part (L_2). the arc-length $s(-l_1 \leqslant s \leqslant l_2)$ is measured along L in the positive sense relative to the flow region, with $s=0$ at C. The shape of the known part L_1 of L is given by the equation

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$$\eta = \eta(s), \quad -l_1 \leqslant s \leqslant 0$$

where η is the angle between the tangent to the contour and the positive direction of the x axis. The distribution of the magnitude of the flow velocity on the unknown part L_2 of L is given

$$\upsilon = f(\sigma), \quad \sigma = s / l_2, \quad 0 \leqslant \sigma \leqslant 1 \tag{1.1}$$

where $f(\sigma)$ is a positive smooth function in the interval [0, 1) (Fig. 1b). The behaviour of $f(\sigma)$ as $\sigma \to 1$ is determined by the angle $\pi\varepsilon (1 < \varepsilon \leqslant 2)$ external to the profile at the trailing edge: $f(1) = \upsilon_*$ if $\varepsilon = 2$ and $f(1) = 0$ if $\varepsilon \neq 2$ (the dashed curve in the neighbourhood of $\sigma = 1$ in Fig. 1b).

Note that in this formulation of the problem the flow branch point A lies on the known part L_1 of L and its position will be determined in due course. The trailing critical point of the flow, according to the Joukowski–Chaplygin postulate, will be the sharp edge B. The leading edge D of the profile lies on L_1, so that the x axis may be directed along the chord BD. In that case α will be the angle of attack.

The quantities ε, υ_∞, l_1, as well as the dimensionless circulation of the flow velocity $\Gamma^* = \Gamma / l_2$ along the profile, are specified (Γ is the dimensional circulation).

It is required to complete the profile and determine its hydrodynamic characteristics l_2, the length of the completed part, and α, the angle of attack, will be determined as part of the solution.

Under the above assumption, a complex flow potential $w(z) = \varphi(x, y) + i\psi(x, y)$ exists, where φ is the velocity potential and ψ is the stream function. Since $\psi = \text{const}$ on an impermeable contour (we will take $\psi = 0$), it follows that on L we have $\upsilon = d\varphi/ds$, i.e. given (1.1), we may assume that the flow velocity distribution is known on the unknown part L_2 of the contour apart from a constant l_2

$$\varphi(\sigma) = l_2 \int_0^\sigma f(\sigma) d\sigma, \quad 0 \leqslant \sigma \leqslant 1 \tag{1.2}$$

Put $\varphi_b = \varphi(1) / l_2$. The region in which w varies, after making a slit along the streamline converging with the trailing edge, is shown in Fig. 2(a). Corresponding points in different planes are denoted by the same letters.

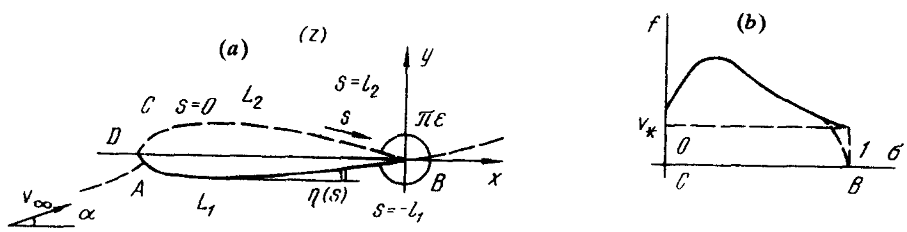

Fig. 1.

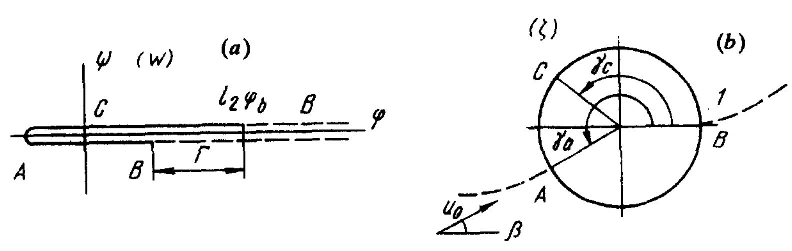

Fig. 2.

2. SOLUTION

The canonical region will be the exterior of the unit circle $|\zeta|\geq 1$, with points corresponding as indicated in Fig. 2(b). To solve the problem we must find a function $z=z(\zeta)$ that maps $|\zeta|\geq 1$ conformally onto the exterior of the required profile, normalized so that $z(1)=0$ and $z(\infty)=\infty$.

Let $\zeta_c=e^{i\gamma_c}$, $\zeta_a=e^{i\gamma_c}$ denote the pre-images of the points C and A in the ζ plane. We know (see, for example, [7, p. 175]) that the function

$$w(\zeta)=u_0\left[\frac{\zeta}{e^{i\beta}}+\frac{e^{i\beta}}{\zeta}\right]-\frac{\Gamma}{2\pi i}\ln\zeta\ln\zeta+K \tag{2.1}$$

is the complex potential of a flow around the circle $|\zeta|\leq 1$ with velocity $u_0e^{i\beta}$ at infinity. Noting that on the boundary of the circle, i.e. at $\zeta=e^{i\gamma}$, the stream function vanishes: $\psi=0$ and $\varphi=0$ at ζ_a, while $w'(\zeta_a)=w'(1)=0$, we obtain

$$K=\frac{\Gamma}{2\pi}\left[\gamma_c-\frac{\cos(\gamma_c-\beta)}{\sin\beta}\right],\quad u_0=\frac{\Gamma}{4\pi\sin\beta},\quad \gamma_a=\pi+2\beta \tag{2.2}$$

Then, setting $\zeta=e^{i\gamma}$, in (2.1) and separating real and imaginary parts, we find that

$$\varphi^*(\gamma)=\operatorname{Re}w(e^{i\gamma})=\frac{\Gamma}{2\pi}\left[\frac{\cos(\gamma-\beta)-\cos(\gamma_c-\beta)}{\sin\beta}-(\gamma-\gamma_c)\right] \tag{2.3}$$

$$\psi^*(\gamma)=\operatorname{Im}w(e^{i\gamma})=0$$

Let us assume that the quantities β, γ_c, as well as a monotone decreasing function $\sigma(\gamma)$, $0\leq\gamma\leq\gamma_c$, defining the correspondence between the points of the circle $\zeta=e^{i\gamma}$ and of the contour L_2 under the conformal mapping $z=z(\zeta)$, are known. The function $\chi(\zeta)=\ln(dw/dz)=\ln\upsilon-i\theta$, where θ is the argument of the velocity vector, is analytic in the region $|\zeta|>1$. It has a logarithmic singularity at the critical points $\zeta=\zeta_a$ and $\zeta=1$. It can be shown that the auxiliary function

$$F(\zeta)=(2-\varepsilon)\ln(1-1/\zeta)+\ln(1-e^{i\gamma_a}/\zeta).$$

has the same singularities as $\chi(\zeta)$. Then the boundary values of the real part of $\Phi(\zeta)=\chi(\zeta)-F(\zeta)$ on the arc $0<\gamma<\gamma_c$ will be

$$\operatorname{Re}\Phi(e^{i\gamma})=S(\gamma)=\ln\left\{f[\sigma(\gamma)]\left[2\cos\left(\frac{\gamma}{2}-\beta\right)\left(2\sin\frac{\gamma}{2}\right)^{2-\varepsilon}\right]^{-1}\right\}$$

We now define a function $\tilde{\Phi}(\zeta)$, regular in $|\zeta|\geq 1$, such that

$$\tilde{S}(\gamma)=\operatorname{Re}\tilde{\Phi}(e^{i\gamma})=\begin{cases}S(\gamma), & 0\leq\gamma\leq\gamma_c\\ S_*(\gamma), & \gamma_c\leq\gamma\leq 2\pi\end{cases}$$

where $S_*(\gamma)$ is an as yet unknown Hölder-continuous function defining the boundary values of the real part of $\tilde{\Phi}(\zeta)$, which is analytic outside the disk $|\zeta|<1$, in such a way that $\tilde{S}(\gamma)$ is a 2π-periodic Hölder-continuous function over the entire interval $0\leq\gamma\leq 2\pi$. Then the function $\tilde{\Phi}(\zeta)=\tilde{S}-i\tilde{\theta}$ admits of an integral representation

$$\tilde{\Phi}(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{S}(\tau)\frac{\zeta + e^{i\tau}}{\zeta - e^{i\tau}}d\tau + ib_0$$

where

$$\tilde{\theta}(\gamma) = -\operatorname{Im}\tilde{\Phi}(e^{i\gamma}) = \frac{1}{2\pi}\int_0^{2\pi} \tilde{S}(\tau)\operatorname{ctg}\frac{\gamma - \tau}{2}d\tau - b_0$$

The constant b_0 is chosen so that $\tilde{\theta}(\gamma_c) = \eta(0)$.

Now, considering the half-plane $\operatorname{Im} t \geqslant 0$ of the complex plane of $t = \zeta + iv$, we define a function $G(t) = i(\tilde{\Phi} - \Phi)$ regular in that half-plane. The region $|\zeta| \geqslant 1$ is mapped onto the half-plane by the function

$$t = i\frac{\zeta + e^{i\gamma_c/2}}{\zeta - e^{i\gamma_c/2}}\operatorname{tg}\frac{\gamma_c}{4} \tag{2.4}$$

To define $G(t)$ we have the following boundary-value problem

$$\operatorname{Im} G = 0 \quad \text{for} \quad |\xi| \geqslant 1, \quad \operatorname{Re} G = g(\xi) = \tilde{\theta}[\gamma(\xi)] - \eta[s(\xi)] \quad \text{for} \quad |\xi| \leqslant 1$$

where $\gamma(\zeta)$ is the inverse of the function $\zeta(\gamma) = \operatorname{Re} t(e^{i\gamma})$, and $s(\xi)$ is the required function, defining the correspondence between the points of L_1 and those of the closed interval $|\zeta| \leqslant 1$. We obtain an integral representation of $G(t)$ using the Keldysh–Sedov formula

$$G(t) = \frac{\sqrt{1 - t^2}}{\pi i}\int_{-1}^{1}\frac{g(\tau)d\tau}{(\tau - t)\sqrt{1 - \tau^2}}$$

Then

$$\chi(\zeta) = \frac{\sqrt{1 - t(\zeta)^2}}{\pi}\int_{-1}^{1}\frac{g(\tau)d\tau}{[\tau - t(\zeta)]\sqrt{1 - \tau^2}} + \tilde{\Phi}(\zeta) + F(\zeta) \tag{2.5}$$

and the required mapping function will be

$$z(\zeta) = \int_1^{\zeta}\frac{dw}{d\zeta}e^{-\chi(\zeta)}d\zeta \tag{2.6}$$

Hence the integral equation for our function $s(\xi)$, $|\xi| \leqslant 1$, will be

$$s(\xi) = \int_0^{\gamma(\xi)}\exp\left\{-\tilde{S}(\gamma) - \frac{\sqrt{1 - \xi(\gamma)^2}}{\pi}\int_{-1}^{1}\frac{g(\tau)d\tau}{[\tau - \xi(\gamma)]\sqrt{1 - \tau^2}}\right\}\left(2\sin\frac{\gamma}{2}\right)^{2-\varepsilon}d\gamma \tag{2.7}$$

This functional equation, which has the form $s(\xi) = \mathbb{P}(s, \xi)$, can be solved numerically using the iterative scheme $s^{(n+1)}(\xi) = \mathbb{P}(s^{(n)}, \xi)$, $n = 1, 2, \ldots$. As an initial approximation one can take the function obtained from (2.7) by putting $g(\xi) = \tilde{\theta}(\xi)$, $|\xi| \leqslant 1$. The process continues until the functions $s^n(\xi)$ obtained in two consecutive iterations are identical to within satisfactory accuracy. Then the coordinates of the profile are computed from (2.6) with $\zeta = e^{i\gamma}$, and one can then find the length l_2 of the completed section. By (2.5), the angle of attack α will be $\alpha = -\operatorname{Im}\chi(\infty)$.

We have thus reduced the problem to determining the parameters β and γ_c function $\sigma = \sigma(\gamma)$, $0 \leqslant \gamma \leqslant \gamma_c$ on L_2. To do this, we compare the increments of the potentials (1.2) and (2.3) along L_2

$$\varphi[\sigma(\gamma)] = \varphi^*(\gamma), \quad 0 \leqslant \gamma \leqslant \gamma_c, \quad 1 \geqslant \sigma \geqslant 0 \tag{2.8}$$

Since A lies in the known section L_1, the functions $\varphi(s)$, $\varphi^*(\gamma)$ are monotone. Consequently, for Eq. (2.8) to be solvable, $\varphi(\sigma)$ and $\varphi^*(\gamma)$ must be equal at the ends of the interval of definition, i.e. $\varphi(1)-\varphi(0)=\varphi^*(0)-\varphi^*(\gamma_c)$. Hence, bearing in mind that $\gamma_a = \pi + 2\beta$ (2.2), we obtain after some algebra

$$f(\gamma_c, \beta) = \varphi_b \tag{2.9}$$

$$\left(f(\gamma_c, \beta) = \frac{\Gamma^*}{2\pi}\left[\frac{\cos\beta - \cos(\gamma_c - \beta)}{\sin\beta} + \gamma_c\right]\right)$$

The function $f(\gamma_c, \beta)$, considered for fixed β, increases monotonically from 0 at $\gamma_c = 0$ to Γ^* at $\gamma_c = \pi + 2\beta$. Consequently, if $\varphi_b > \Gamma^*$, formula (2.9) defines a single-valued function $\beta = \beta(\gamma_c)$.

Thus, we have a free parameter γ_c. To find it, we recall that the velocity υ_∞ is given. But this quantity is determined in the course of the solution: it follows from (2.5) that as $\zeta \to \infty$

$$\upsilon_\infty = \exp\left\{ \operatorname{Re} \frac{\sqrt{1-t_\infty^2}}{\pi} \int_{-1}^{1} \frac{g(\tau)d\tau}{(\tau - t_\infty)\sqrt{1-\tau^2}} + \frac{1}{2\pi}\int_0^{2\pi} \tilde{S}(\tau)d\tau \right\} \tag{2.10}$$

where, by (2.4)

$$t_\infty = t(\infty) = i\,\mathrm{tg}(\gamma_c / 4)$$

Equation (2.10) may be used to determine γ_c.

The free parameter γ_c may also be used to ensure that υ_∞ is as large as possible. This method of choosing γ_c will be considered below in greater detail in the context of our numerical computations.

3. THE CONDITIONS OF SOLVABILITY AND THE CONSTRUCTION OF THE QUASISOLUTION

By the conditions of solvability for an inverse boundary-value problem of aerodynamics one usually means conditions under which the required wing profile is closed and the condition that the prescribed velocity υ_∞ coincide with that determined during the course of the solution. In our problem the last condition may be satisfied by a suitable choice of the parameter γ_c. The conditions for the contour to be closed are

$$\int_0^{2\pi} S(\tau)e^{i\tau}d\tau = \pi(\varepsilon - 1) \tag{3.1}$$

A quasisolution may be constructed by the technique described in [8], that is, minimum correction of the function $S(\gamma)$, e.g. on the contour L_2, i.e. $0 \leqslant \gamma \leqslant \gamma_c$. The result is a new function $S^*(\gamma)$, which satisfies condition (3.1). When that is done, however, the contour L_1 corresponding to $S^*(\gamma)$ need not have the prescribed shape. In order to ensure closure of the curve and at the same time obtain the prescribed L_1, we proceed as follows. Taking $S^*(\gamma)$ as the next approximation for $\tilde{S}(\gamma)$, we repeat the procedures described above to determine $s(\xi)$ and construct a quasi-solution, thus getting a new approximation to $S^*(\gamma)$. This procedure is now repeated until a profile with a closed contour is obtained. By construction, this profile will have the same prescribed part L_2 of the contour and a velocity distribution diverging only minimally from that prescribed on L_2.

4. NUMERICAL COMPUTATIONS

Figure 3 shows the results of a test computation and a computation whose purpose was to modify the upper section of a profile in such a way as to liquidate the velocity peak in the vicinity of the trailing edge.

The test computation proceeded as follows. We considered flow around a 10% Zhukovskii rudder (curve 1, solid below and dashed above) with $v_\infty = 1$ at an angle of attack $\alpha = 5°$. Analytical formulae were used to calculate the distribution $v(\tilde{s})$ (curve 3, dashed along the lower surface and solid along the upper) and the numbers $\Gamma^* = 0.31$, $C_y = 0.60$. The argument $\tilde{s}$ used to describe the computations was related to s by the formula $\tilde{s} = s + l_1$. Solving the mixed inverse problem, we took as initial approximations the lower profile surface (the solid part of curve 1) and velocity distribution $v(\tilde{s})$ over the upper surface (the solid part of curve 3). By the symmetry of the profile $\gamma_c = \pi$. With these data, the technique described above produced a profile practically identical with the initial Zhukovskii rudder. To estimate the computation error in the coordinates, we computed $T = \max_\zeta |z^*(\zeta) - z(\zeta)|$, where $z^*(\zeta)$ is an analytical solution for the Zhukovskii rubber and $z(\zeta)$ is the function obtained in the test computation. The result was $T \leqslant 0.001$. The coefficient $C_y = 0.60$ coincided up to the second place with the initial value.

It can be seen from Fig. 3(b) that the Zhukovskii rudder has a velocity peak on its upper surface, with $v_{max}/v_\infty = 1.81$, which may cause the formation of a cavitation cavern in that part of the profile. To improve the hydrodynamic properties of the profile, the peak was truncated at the level $v/v_\infty = 1.50$. The velocity distribution thus corrected was used as the initial approximation in constructing a modified profile with the same lower surface as the Zhukovskii rudder. As in the test computation, we set $\gamma_c = \pi$. The result was a profile (the dash–dot contour 2) with $t = 10.5\%$, $C_y = 0.60$, in a flow with $v_\infty = 0.95$ and angle of attack $\alpha = 5.01°$. The velocity distribution over its surface is represented in Fig. 3b) by the dash–dot curve 4. The velocity peak for this profile was found to be $v/v_\infty = 1.65$, i.e. significantly less than the initial value of 1.81. However, the magnitude of v_∞ was reduced, so that the maximum relative velocity v/v_∞ exceeded 1.50.

To avoid this situation, we considered the following example, attempting a second modification of the upper surface of the same Zhukovskii rudder (the dashed curve 1 in Fig. 4a), given $v(\tilde{s})$ with a truncated velocity peak at the level $v/v_\infty = 1.5$, with the value of γ_c, however, not prescribed in advance but determined from the condition $v_\infty = 1$. the result was $\gamma_c = 3.17$. The profile thus constructed, with $t = 11.5\%$, is represented by the solid curve 2 in the same figure. It has $C_y = 0.60$ and $\alpha = 4.52°$. With these parameters of the incident flow $v(\tilde{s})$ we obtained the solid curve 3 in Fig. 4(b). For comparison, the figure also shows the initial distribution $v(\tilde{s})$ for the Zhukovskii rudder (the dashed curve 4). It is clear that modification of the upper surface made it possible not only to "truncate" the velocity peak down to 1.54 but also only slightly to exceed the given ratio v_{max}/v_∞.

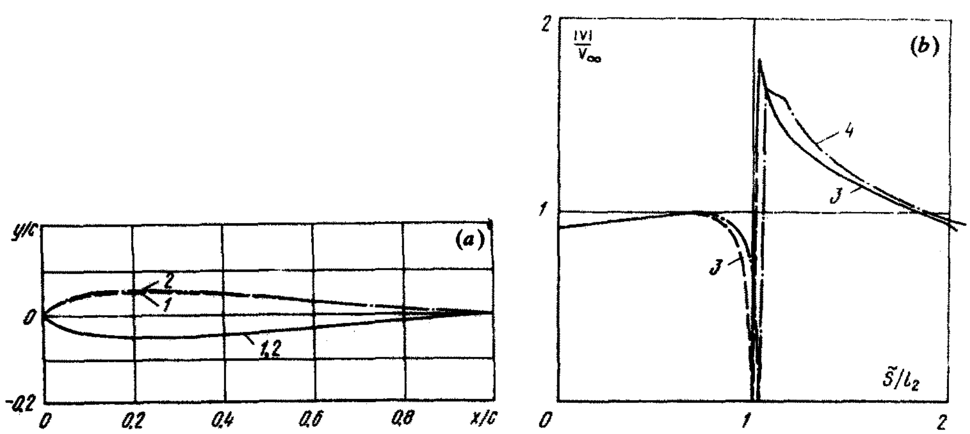

Fig. 3.

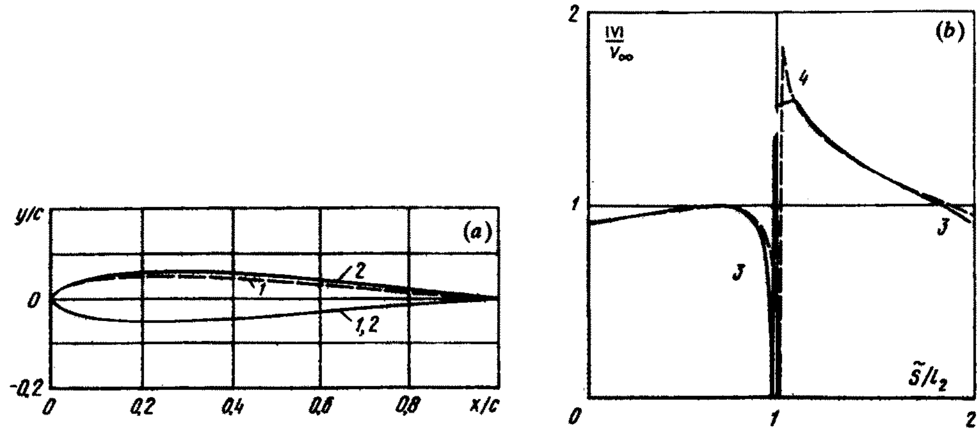

Fig. 4.

Modification of the suction side of the profile, aimed at achieving maximum velocity of the incident flow, is of practical interest. To that end one can use the free parameter γ_c. A series of computations showed that $v_\infty(\gamma_c)$ is a monotone increasing function of γ_c, but as the latter increases there is a corresponding decrease in the rate of convergence of the method for constructing close profiles; for sufficiently large γ_c the procedure in fact no longer converges. If the initial data are, as in the previous examples, the lower surface of a Zhukovskii rudder (the dashed curve 1 in Fig. 5a) and a distribution $v(\tilde{s})$ with velocity peak truncated at a level of 1.5 (the dashed curve 3 in Fig. 5(b), the limiting value of γ_c turns out to be close to 3.23. The corresponding profile (the third modification) with $t = 12.35\%$ is represented by the solid curve 2 in Fig. 2(a). It has $C_y = 0.61$ when $\alpha = -1.97°$ and $v_\infty = 1.16$. The distribution $v(\tilde{s})$ is represented by the solid curve 4 in Fig. 5(b).

Figure 6 compares the results of all three modifications, with γ_c: (1) prescribed in advance, (2) determined from the condition $v_\infty = 1$, and (3) determined from the condition that v_∞ should be a maximum. The solid curves 1 and 4 represent the corresponding profile and velocity distribution $v(\tilde{s})$ in the third modification, the dashed curves 2 and 5 represent the same data in the second modification, and the dash–dot curves—in the first. Table 1 compares the geometrical and hydrodynamic characteristics of all the profiles considered.

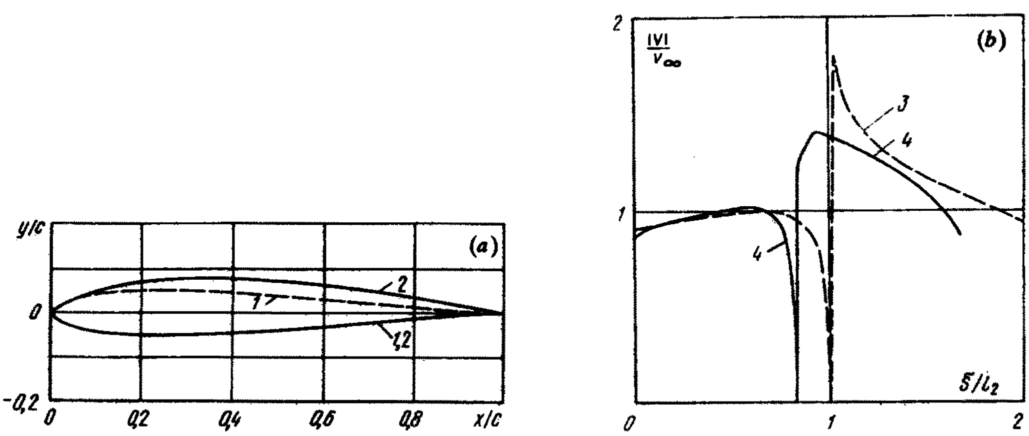

Fig. 5.

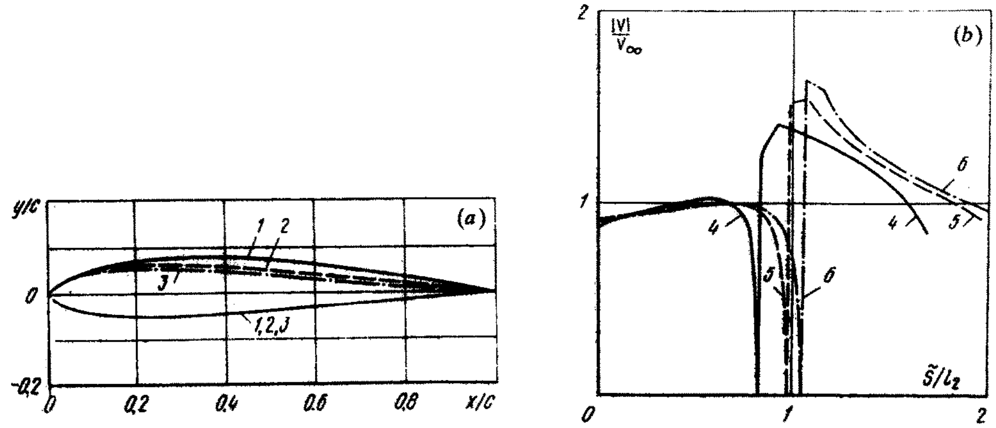

Fig. 6.

Table 1

Problem	γ_c	v_∞	α°	v_{max}/v_∞	C_y	$t\%$
Zhukovskii rudder	3,14	1,00	5,00	1,81	0,60	10,0
First modification	3,14	0,95	5,01	1,65	0,60	10,5
Second modification	3,17	1,00	4,52	1,54	0,60	11,5
Third modification	3,23	1,16	−1,97	1,41	0,61	12,4

The results clearly show that, using the technique described here, it has been possible to achieve a significant reduction in the maximum velocity on the upper surface of the profile (the second and third modifications). By construction, the shape of the lower surface remains unchanged and the velocity of the incident flow is either equal to its prescribed value or maximized. An interesting point is that the value of C_y has changed only slightly throughout all the modifications.

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